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Spacetime algebraic skeleton

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The cosmological constant Λ, which has seemingly dominated the primaeval Universe evolution and to which recent data attribute a significant present-time value, is shown to have an algebraic content: it is essentially an eigenvalue of a Casimir invariant of the Lorentz group which acts on every tangent space. This is found in the context of de Sitter spacetimes but, as every spacetime is a 4-manifold with Minkowski tangent spaces, the result suggests the existence of a "skeleton" algebraic structure underlying the geometry of general physical spacetimes. Different spacetimes come from the "fleshening" of that structure by different tetrad fields. Tetrad fields, which provide the interface between spacetime proper and its tangent spaces, exhibit to the most the fundamental role of the Lorentz group in Riemannian spacetimes, a role which is obscured in the more usual metric formalism.

I. INTRODUCTION

Until rather recently the Universe was believed to follow a Friedmann solution of Einstein's equations, which comes from using as a source a gas of matter and radiation [1, 2]. The horizon problem forced a change in this view: at the first stages, the Universe should be expanding at a much larger rate than Friedmann's prediction, possibly as it would be if the solution were a de Sitter spacetime. Such a spacetime is a solution without any source but with a universal curvature encapsulated in a cosmological constant. Its great interest comes from de Sitter's finding that Einstein's equations without any source have such non-flat solutions. Flat Minkowski space is a particular case, with vanishing cosmological constant. The overall picture would be that of an initial de Sitter Universe which changes to a Friedmann Universe at an early stage [3].

Recent data point, however, to a significant value for the cosmological constant even today [4, 5]. This effect is usually simply added by hand to the Friedmann description. The new picture is that of a Universe which begins as a de Sitter spacetime and then changes to the solution describing the present-day state — a Friedmann solution with a significant de Sitter remnant. One might wonder whether the Friedmann "component" could not be seen as a perturbation of a spacetime which remains basically of the de Sitter type.

De Sitter spacetimes have a special characteristic, maximal symmetry. They have ten Killing vectors, the maximum possible for 4-dimensional spaces. The above cosmological picture can be rephrased: the Universe is initially empty, with a maximal symmetry which is *partially* broken when matter and radiation somehow turn up. All this calls for a reappraisal of de Sitter spaces, looking as deep as possible in its foundations. We intend here to examine the algebraic structure behind their geometrical make-up.

Section II sums up the tetrad formalism, emphasis being given to the manifestations of the Lorentz group — in particular, to the fact that the Christoffel-Levi-Civita connection reveals itself as a Lorentz connection when looked at from any tetrad frame. Section III introduces de Sitter spacetimes through stereographic coordinates, which suggest a specially simple tetrad frame in which the subsequent topics are considered. The algebra of bundle vector fields, in the case a base for the de Sitter Lie algebra, is recalled in section IV. The cosmological constant is examined through its relation to curvature in section V, where it is shown to be ultimately an eigenvalue of a Casimir invariant operator of the Lorentz group.

II. SPACETIME: RÔLE OF THE LORENTZ GROUP

A spacetime S is a four-dimensional differentiable manifold whose tangent space is, at each point, a Minkowski space. In bundle language, the tangent bundle TS on any spacetime has the Minkowski space M as the typical fiber. The linkage between the Minkowski typical fiber and the spaces tangent to spacetime is provided by tetrad fields. A

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tetrad field will solder a copy of M as tangent space T_pS at each point $p \in S$. The tangent bundle is associated to the principal bundle BM of tetrad frames. Connections are first defined on BM and then induced on each associated bundle, taking into account the respective representation of the structure (Lorentz) group. A spacetime appears then as the quotient of BM by the Lorentz group. The de Sitter spacetimes are special because, in their cases, the bundle BM of Lorentzian frames are Lie groups, just the corresponding de Sitter groups.

Consider on the typical fiber M the standard canonical vector basis $\{K_0 = (1,0,0,0), K_1 = (0,1,0,0), K_2 = (0,0,1,0), K_3 = (0,0,0,1)\}$. With the convention $a, b, \ldots = 0,1,2,3$, each K_a is given by the entries $(K_a)_b = \delta_{ab}$. A tetrad field e will be a mapping e :M$\rightarrow TS$, $e(K_a) = e_a$. This set of four vectors e_a will constitute a local vector basis on S: given a point $p \in S$, and around it an euclidean open set U, the e_a's will constitute a vector basis not only for the space T_pS tangent to S at p but also for all the T_qS with $q \in U$. The extension from p to U is warranted by the differentiable structure [6]. The dual forms ω^b, such that $\omega^b(e_a) = \delta^b_a$, will constitute a covector basis on the cotangent space at p, T^*_pS. In most applications the open set U is a coordinate neighborhood. Each coordinate system $\{x^\mu\}$ on U will define a "natural" (or "coordinate") vector basis $\{\partial_\mu = \frac{\partial}{\partial x^\mu}\}$ with its concomitant covector dual basis $\{dx^\mu\}$. This is a very particular and convenient tetrad field, frequently called a "trivial" tetrad, given by $e(K_a) = \delta^\mu_a \partial_\mu$. It is usual to fix a coordinate system around each point p from the start, and in this sense this basis is indeed "natural". Once such a coordinate system is fixed, with its accompanying pair of trivial bases, any other tetrad field like the above generic $\{e_a\}$ and its dual $\{\omega^b\}$ can be written as [8, 9]

$$e_a = h_a{}^\mu \partial_\mu \text{ and } \omega^b = h^b{}_\nu dx^\nu, \tag{1}$$

with the conditions

$$h^b{}_\mu h_a{}^\mu = \delta^b_a \text{ and } h^a{}_\mu h_a{}^\nu = \delta^\nu_\mu. \tag{2}$$

The components $h_a{}^\mu(x)$ have one label referring to the typical Minkowski space and another in tangent spacetime. We are using latin letters $(a, b, \ldots = 0,1,2,3)$ to label components on M and the greek alphabet for spacetime indices. We shall call the first "Minkowski indices" (or "tetrad indices"), the latter "Riemann indices". In their Minkowski labels the tetrads change under Lorentz transformations according to

$$h^{a'}{}_\mu(x) = \Lambda^{a'}{}_c(x)\; h^c{}_\mu(x). \tag{3}$$

$\Lambda^{a'}{}_b$ is constant on each tangent space T_pS, but will depend on the point p of spacetime, indicated above by its coordinates $x = \{x^\mu\}$. Contracting the last expression with $h_b{}^\mu(x)$, we obtain the Lorentz transformation in terms of the initial and final tetrad basis:

$$\Lambda^{a'}{}_b(x) = h^{a'}{}_\mu(x) h_b{}^\mu(x). \tag{4}$$

Equation (3) says that each tetrad component behaves, on each Minkowski fiber, as a Lorentz vector. More precisely, $h^a = h^a{}_\mu dx^\mu$ transforms according to the covector representation of the Lorentz group. A Lorentz covector has components changing, under a transformation with parameters α^{cd}, according to

$$\phi^{a'} = \Lambda^{a'}{}_b\; \phi^b = \left(exp[\tfrac{1}{2}\alpha^{cd} J_{cd}]\right)^{a'}{}_b\; \phi^b. \tag{5}$$

Here, each J_{cd} is a 4×4 matrix representing one of the Lorentz group generators:

$$[J_{cd}]^{a'}{}_b = \eta_{db}\; \delta^{a'}_c - \eta_{cb}\; \delta^{a'}_d\; . \tag{6}$$

We use the convention $\eta = (\eta_{ab}) = \mathrm{diag}(1, -1, -1, -1)$ for the Lorentz metric.

A tetrad field converts tensors on M into tensors on spacetime. For example, they will produce a field on spacetime out of a vector in Minkowski space by

$$\phi^\mu(x) = h_a{}^\mu(x) \phi^a\; . \tag{7}$$

As the Minkowski indices are contracted, $\phi^\mu(x)$ is Lorentz-invariant. Another case of tensor transmutation is $\Lambda^\mu{}_\nu(x) = h_{a'}{}^\mu(x)\; \Lambda^{a'}{}_b(x)\; h^b{}_\nu(x) = \delta^\mu_\nu$, which shows that there is no Lorentz transformation on spacetime itself. The Lorentz metric η_{ab} is transmuted into the Riemannian metric

$$g_{\mu\nu}(x) = \eta_{ab}\; h^a{}_\mu(x)\; h^b{}_\nu(x). \tag{8}$$

Different tetrad fields transmute η_{ab} into different spacetime metrics. Of course, also $g_{\mu\nu}(x)$ is Lorentz-invariant. The Lorentz group is concealed by the transmutations, and remains out of sight in the usual metric formalism. The members of a general tetrad field will satisfy a set of commutation rules

$$[e_a, e_b] = c^c{}_{ab}\, e_c. \tag{9}$$

The structure coefficients measure its anholonomy and are given by

$$c^c{}_{ab} = [e_a(h_b{}^\mu) - e_b(h_a{}^\mu)]h^c{}_\mu\ . \tag{10}$$

If $\{e_a\}$ is holonomous ($c^c{}_{ab} = 0$), then $\omega^a = dx^a$ for some coordinate system $\{x^a\}$ and $dx^{a'} = \Lambda^{a'}{}_b dx^b$. Expression (8) would then give just the Lorentz metric written in another system of coordinates. In the general Riemannian case, $\{e_a\}$ is anholonomic.

A procedure converse to that followed above is more usual: given the metric, the tetrad field is defined by (8). This is natural when the metric is known *a priori* or, as in General Relativity, as the solution of a dynamical equation. Einstein's equations fix the metric and determine the tetrads up to Lorentz transformations.

Connections have a special behavior under transmutation. A general Lorentz connection has the form

$$\Gamma = \tfrac{1}{2} J_{ab}\ \Gamma^{ab}{}_\mu\ dx^\mu = \tfrac{1}{2} J_{ab}\Gamma^{ab}{}_\mu\ h_c{}^\mu\omega^c = \tfrac{1}{2} J_{ab}\Gamma^{ab}{}_c\ \omega^c \tag{11}$$

(the factors 1/2 account for double counting). The first two indices in the components $\Gamma^a{}_{b\mu}$ are "algebraic" and the last a vector index. Under a Lorentz transformation $\Lambda^{a'}{}_a = h^{a'}{}_\lambda h^\lambda{}_a$,

$$\Gamma^{a'}{}_{b'\nu} = \Lambda^{a'}{}_a\ \Gamma^a{}_{b\nu}\ (\Lambda^{-1})^b{}_{b'} + \Lambda^{a'}{}_c\ \partial_\nu\ (\Lambda^{-1})^c{}_{b'}. \tag{12}$$

The Levi-Civita connection is

$$\Gamma^\lambda{}_{\nu\mu} = h_b{}^\lambda \partial_\mu h^b{}_\nu + h_a{}^\lambda \Gamma^a{}_{b\mu} h^b{}_\nu \tag{13}$$

is Lorentz-invariant. Components $\Gamma^a{}_{b\mu}$ and $\Gamma^\lambda{}_{\nu\mu}$ refer to different spaces, but represent he same connection. The last expression describes how Γ changes when the Minkowski indices are transmuted into Riemann indices. Components $\Gamma^\lambda{}_{\mu\nu}$ will, under a base transformation $B^{\lambda'}{}_\lambda = h_a{}^{\lambda'} h^a{}_\lambda$, change according to

$$\Gamma^{\lambda'}{}_{\mu'\nu} = B^{\lambda'}{}_\lambda \Gamma^\lambda{}_{\mu\nu} B^\mu{}_{\mu'} + B^{\lambda'}{}_\lambda \partial_\nu B^\lambda{}_{\mu'}.$$

A connection acts on a field through the generators of the Lorentz representation it belongs. On a spinor, for example, through the spinor generators — which is the origin of the name "spin connection" frequently given to $\Gamma^a{}_{b\mu}$. On a vector U, it acts with the generators of Eq.(6):

$$\Gamma^a{}_{c\mu} U^c = \tfrac{1}{2}\ \Gamma^{ef}{}_\mu (J_{ef})^a{}_c U^c.$$

A good illustration of this action turns up when we say that a vector "stays the same" by parallel transport. Consider, for example, along a curve with parameter u, the covariant derivative of its tangent velocity U:

$$\frac{\nabla U_\lambda}{\nabla u} = \frac{dU_\lambda}{du} - \Gamma^\mu{}_{\lambda\nu} U_\mu U^\nu = \frac{dU_\lambda}{du} + U_\nu \partial_\lambda U^\nu = U^\nu(\partial_\nu U_\lambda - \partial_\lambda U_\nu).$$

Seen from the tetrad frame $e_a = h_a{}^\mu \partial_\mu$, this leads to

$$\frac{\nabla U_a}{\nabla u} = \frac{\nabla h_a{}^\lambda U_\lambda}{\nabla u} = U_c\ h^c{}_\lambda \frac{\overset{\circ}{\nabla} h_a{}^\lambda}{\nabla u} + U^b[e_b(U_a) - e_a(U_b) + c^e{}_{ab} U_e].$$

As $h^c{}_\lambda \frac{\overset{\circ}{\nabla} h_a{}^\lambda}{\nabla u} = \Gamma^c{}_{ab} U^b$ and $\Gamma_{cab} U^b U^c = -\ \frac{1}{2}\ [c_{cab} + c_{abc} + c_{bac}] U^b U^c = -\ c_{cab} U^b U^c$,

$$\frac{\nabla U_a}{\nabla u} = U^b[\delta^c_b \delta^d_a - \delta^c_a \delta^d_b] e_c(U_d) + (c^e{}_{ab} + \Gamma^e{}_{ab}) U_e] = U^b (J_{ba})^{cd} e_c(U_d).$$

Along a general curve, the velocity U is continuosly changed by infinitesimal Lorentz transformations. These transformations vanish along a geodesic.

III. DE SITTER SPACETIMES

There are only two kinds of de Sitter spacetimes [2]: that usually called de Sitter spacetime proper, and the so-called anti-de Sitter spacetime. Their respective groups of motions are the two de Sitter groups $SO(4,1)$ and $SO(3,2)$. If the curvature is made to tend to zero, both groups reduce to the Poincaré group $SO(3,1) \oslash T^{3,1}$ by a Inonü-Wigner contraction [10] and both spaces reduce to Minkowski space.

These three groups have much in common [11, 12]: their manifolds are fiber bundles structured by the Lorentz group ($SO(3,1)$ or its covering). Base spaces are the respective spacetimes. Thus, the Poincaré group is the complete space of a principal bundle with Minkowski space as base space, and each de Sitter group is the complete space of a principal bundle with the corresponding de Sitter space as base space. The group manifolds coincide, in these cases, with the bundles of Lorentzian frames on the respective spacetimes. The latter are the quotients $SO(4,1)/SO(3,1)$, $SO(3,2)/SO(3,1)$ and $(SO(3,1) \oslash T^{3,1})/SO(3,1)$.

We shall introduce the de Sitter and the anti-de Sitter spacetimes as hypersurfaces in the pseudo–Euclidean spacetimes $\mathbf{E}^{4,1}$ and $\mathbf{E}^{3,2}$, inclusions whose points in Cartesian coordinates $(\xi) = (\xi^0, \xi^1, \xi^2, \xi^3, \xi^4)$ satisfy, respectively,

$$\begin{aligned} (\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 - (\xi^4)^2 &= \eta_{ab}\,\xi^a\xi^b - (\xi^4)^2 = -\,L^2\,; \\ (\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 + (\xi^4)^2 &= \eta_{ab}\,\xi^a\xi^b + (\xi^4)^2 = \quad L^2\,. \end{aligned}$$

L is a constant "pseudo-radius". The sign notation $\eta_{44} = s$ allows these conditions to be put together as

$$s\;\eta_{ab}\,\xi^a\xi^b + \left(\xi^4\right)^2 = L^2\,.$$

We now change to stereographic coordinates $\{x^\mu\}$ in 4–dimensional space, which are given by

$$x^\mu = h_a{}^\mu\,\xi^a\,, \tag{14}$$

where a specially simple tetrad field turns up:

$$h_a{}^\mu = \tfrac{1}{n}\,\delta_a^\mu\,, \quad \text{where} \quad n = \tfrac{1}{2}\left(1 - \tfrac{\xi^4}{L}\right) = \tfrac{1}{1+s\sigma^2/4L^2} \quad \text{and} \quad \sigma^2 = \eta_{ab}\,\delta^a{}_\mu\delta^b{}_\nu\,x^\mu x^\nu. \tag{15}$$

Calculating the line element on the hypersurfaces, we find $ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu$, where

$$g_{\mu\nu} = h^a{}_\mu h^b{}_\nu \eta_{ab} = n^2\delta^a_\mu\delta^b_\nu\eta_{ab}\,. \tag{16}$$

The most natural tetrad base and its dual are

$$e_a = h_a{}^\mu\partial_\mu = \tfrac{1}{n}\,\delta_a{}^\mu\partial_\mu \quad \text{and} \quad \omega^a = h^a{}_\mu dx^\mu = n\delta^a{}_\mu dx^\mu\,. \tag{17}$$

Equation (16) tells us that de Sitter spacetimes are conformally flat [1] with conformal factor $n^2(x)$. The Christoffel symbols corresponding to the above metric are

$$\Gamma^\lambda{}_{\mu\nu} = \left[\delta^\lambda{}_\mu\delta^\sigma{}_\nu + \delta^\lambda{}_\nu\delta^\sigma{}_\mu - g_{\mu\nu}g^{\lambda\sigma}\right]\partial_\sigma \ln n\,, \tag{18}$$

leading to the Riemann and Ricci tensor components

$$R^\alpha{}_{\beta\rho\sigma} = -\,\frac{s}{L^2}\left[\delta^\alpha_\sigma g_{\beta\rho} - \delta^\alpha_\rho g_{\beta\sigma}\right], \quad R_{\mu\nu} = \frac{3s}{L^2}\,g_{\mu\nu}, \tag{19}$$

and to the scalar curvature

$$R = 12\,\frac{s}{L^2}\,. \tag{20}$$

IV. THE ALGEBRA OF VECTOR FIELDS

Distinct de Sitter spacetimes (including Minkowski) differ by their curvatures, and bundles are the natural place to study connections and their curvatures. The local geometrical properties are specially evident in the space tangent to the bundle. As the groups coincide with the bundles, the vertical tangent vector fields can be directly related to the generators. Thus, from a local point of view, that difference in curvature should be reflected in the corresponding Lie algebras. Besides the Lorentz generators $\{J_{ab}\}$ of Eq.(6), we shall use the generators $\{T_c\}$ of spacetime translations.

Given the Lorentz metric η_{ab}, consider the three sets of structure constants:

$$f^{(ef)}{}_{(ab)(cd)} = \eta_{bc}\delta^e{}_a\delta^f{}_d + \eta_{ad}\delta^e{}_b\delta^f{}_c - \eta_{bd}\delta^e{}_a\delta^f{}_c - \eta_{ac}\delta^e{}_b\delta^f{}_d \tag{21}$$

$$f^{(e)}{}_{(ab)(c)} = \eta_{cb}\delta^e{}_a - \eta_{ca}\delta^e{}_b \tag{22}$$

$$f^{(ef)}{}_{(a)(b)} = \left(\delta^f{}_a\delta^e{}_b - \delta^e{}_a\delta^f{}_b\right) = -\, f^e{}_{(ab)}{}^f\ . \tag{23}$$

The Lie algebras of the de Sitter groups are then given by

$$[J_{cd}, J_{ef}] = \tfrac{1}{2} f^{(ab)}{}_{(cd)(ef)}\, J_{ab} \tag{24}$$
$$[J_{cd}, T_e] = f^{(a)}{}_{(cd)(e)}\, T_a \tag{25}$$
$$[T_c, T_e] = \tfrac{1}{2}\, s\, f^{(ab)}{}_{(c)(e)}\, J_{ab}\ . \tag{26}$$

The difference with respect to the Poincaré group lies in the non-commutativity of de Sitter translations. We can use Eq.(26) all the way long, putting $s = 0$ in the Poincaré case. Notice the dimensional anomaly of this equation: translation generators have dimension length^{-1}, while Lorentz generators are dimensionless. Though acceptable as an abstract relation, it must be corrected in any physical application. This has been done by the factors of L^2 which turned up in the previous section.

The direct-product character of the bundle is recovered by using a tetrad to take the translations "down" to spacetime. We shall have ∂_μ or $\nabla_\mu = h^a{}_\mu T_a$, depending on whether we are interested in a common derivative or a covariant one. In both cases, the middle commutators (25) vanish. For example, $[J_{cd}, \partial_\mu] = 0$ because $h^a{}_\mu$, with a covector index a, transforms just in the opposite way to T_a, for which a is a vector index: $J_{cd}(h^a{}_\mu) = -\, f^{(a)}{}_{(cd)(e)} h^e{}_\mu$.

It can be verified from (22) that the Lorentz metric satifies

$$\eta_{de}\, f^{(e)}{}_{(ab)(c)} + \eta_{ce}\, f^{(e)}{}_{(ab)(d)} = 0\ . \tag{27}$$

This provides a precise meaning to raising and lowering indices of the structure constants and their derived quantities. If we accept to lower the entry indices with the Lorentz metric, then the matrices $f_{(ab)}$ with entries $(f_{(ab)})_{ec} = f_{(e)(ab)(c)}$ are antisymmetric. Applying tetrads we find that also $g_{\mu\nu}\, f^{(\nu)}{}_{(ab)(\lambda)} + g_{\lambda\nu}\, f^{(\nu)}{}_{(ab)(\mu)} = 0$. Metrics satisfying this type of condition, the same as Eq.(27), are invariant metrics of the algebra [13].

Additional information can be obtained from the Jacobi identities. There are four types, according to the numbers of generators of each kind involved: $[L,[L,L]]$, $[L,[L,T]]$, $[L,[T,T]]$ and $[T,[T,T]]$. Some of them give no information at all in the $s = 0$ Poincaré case. The Lorentz sector constitutes a closed subalgebra, so that type $[L,[L,L]]$ has nothing to say on the translations. The $[L,[L,T]]$ identity

$$[J_{ab}, [J_{cd}, T_e]] + [T_e, [J_{ab}, J_{cd}]] + [J_{cd}, [T_e, J_{ab}]] = 0$$

gives the condition

$$f^{(h)}{}_{(ab)(f)} f^{(f)}{}_{(cd)(e)} - f^{(h)}{}_{(cd)(f)} f^{(f)}{}_{(ab)(e)} = \tfrac{1}{2} f^{(f)}{}_{(ab)(cd)} f^{(h)}{}_{(gf)(e)}.$$

Comparison with Eq.(24) tells that each generator J_{ab} can be represented by a 4×4 matrix with elements

$$\left(J_{ab}\right)^c{}_e = f^{(c)}{}_{(ab)(e)}. \tag{28}$$

This corresponds precisely to the vector representation (6) of the Lorentz group and gives the meaning of the second lines in the commutation tables above: the translation generators are Lorentzian vectors. By Eq.(27), the lowered-index matrices are antisymmetric.

V. THE COSMOLOGICAL CONSTANT

The action of J_{ab} on the vectors T_c is summed up in Eq.(25): through the commutator in the left-hand side, by matrix multiplication in the right-hand side. This vector representation will have an important role in what follows. Indeed, the $[T,[T,T]]$ Jacobi identity gives

$$s\ f^{(h)}{}_{(ef)[(a)} f^{(ef)}{}_{(b)(c)]} = 0,$$

where $[abc]$ stands for the summation over all cyclic permutations of the included indices. The left factor is a matrix element $(J_{ef})^e{}_a$ for a Lorentz generator in the vector representation. We can interpret each right factor as a component of some matrix along J_{ef}: introduce the matrices F_{bc} with dimension of curvature (L^{-2}) and components $f^{(ef)}{}_{(b)(c)}$ along each J_{ef}:

$$F_{ab} = \tfrac{s}{2L^2} J_{ef} f^{(ef)}{}_{(a)(b)}. \tag{29}$$

Such matrices will have entries $R^c{}_{dab} = (F_{ab})^c{}_d$, or

$$R^c{}_{dab} = \tfrac{s}{2L^2} f^{(c)}{}_{(ef)(d)} f^{(ef)}{}_{(a)(b)}. \tag{30}$$

Combined with metric invariance (27), the antisymmetry of $R^e{}_{abc}$ in the first two indices is obtained. The $[T,[T,T]]$ Jacobi identity will then say that

$$R^c{}_{[dab]} = 0. \tag{31}$$

This is a suggestive expression, formally identical to the vanishing–torsion Bianchi identity ("cyclic property") for the Riemann tensor. This means that the Bianchi identity is the geometrical version of the Jacobi identity, or the Jacobi identity is the algebraic version of the Bianchi identity.

Adding a connection will correspond to a non-trivial extension [14, 15] of the translation algebra, that is, from $s=0$ to $s \neq 0$ in Eq.(26). We should perhaps change the notation acoording to $T_a \Rightarrow D_a$, as the "extended" translation is actually a covariant derivative

$$D_c = e_c + \tfrac{1}{2} A^{ab}{}_c J_{ab}. \tag{32}$$

The $A^{ab}{}_c$'s are the connection components in base $\{h_a\}$ and the J_{ab}'s are the generators in the corresponding representation. Applied to a vector field, for example, their commutator is, by Eq.(26),

$$[D_a, D_b] V^e = \tfrac{1}{2} s f^{(cd)}{}_{(a)(b)} (J_{cd})^e{}_f V^f = \tfrac{1}{2} s f^{(cd)}{}_{(a)(b)} f^{(e)}{}_{(cd)(f)} V^f. \tag{33}$$

Notice the difference in the meaning of the indices: the pair (ab) in $A^{ab}{}_c J_{ab}$ indicates the component of matrix A along J_{ab}. The usual indices in the standard notation signify matrix entries, $A^{ab}{}_c = \frac{1}{2} A^{cd}{}_c (J_{cd})^{ab}$. Our notation has been chosen to make them coincide. Comparison of (33) with the standard formula

$$V^c{}_{;a;b} - V^c{}_{;b;a} = R^c{}_{dab} V^d$$

shows that $R^c{}_{dab}$ as introduced in Eq.(30) is indeed the curvature tensor. We have then a strong relationship with the de Sitter algebra. Using Eqs.(23) and (27), we find that the Ricci tensor is a matrix element:

$$R_{ab} = -\tfrac{s}{2L^2} f_{(b)}{}^{(cd)}{}_{(e)} f^{(e)}{}_{(cd)(a)} = -\tfrac{s}{L^2} (J^2)_{ab} \tag{34}$$

with $J^2 = \frac{1}{2} J^{(ab)} J_{(ab)}$ a Casimir operator of the Lorentz group. The scalar curvature is consequently its trace:

$$R = -\tfrac{s}{L^2} \mathrm{tr} J^2. \tag{35}$$

As $[J^2]^a{}_b = -3\delta^a{}_b$ from Eq.(23), we recover indeed the "geometrical" expression of Eq.(20). On the other hand Einstein's equation for the case,

$$R^a{}_b - \tfrac{1}{2} \delta^a{}_b R = \Lambda \delta^a{}_b, \tag{36}$$

acquires an algebraic version:

$$-\tfrac{s}{L^2} \left[J^2 - \tfrac{1}{2} (\mathrm{tr} J^2) I \right] = \Lambda\ I. \tag{37}$$

We find from Eq.(36) that the cosmological constant is $\Lambda = -R/4$, or

$$\Lambda = \tfrac{s}{4L^2} \mathrm{tr} J^2 = \tfrac{s}{L^2}\ \times \text{the eigenvalue of } J^2 \text{ in the vector representation}. \tag{38}$$

It is, consequently, a purely algebraic invariant of the Lorentz group. Notice that it is the numerical value of the parameter L which is measured experimentally. De Sitter spaces are, of course, unique both because of their conformal character and because their Lorentz bundle is the manifold of a group, which is furthermore semisimple. They have one further exceptional feature: as symmetric homogeneous spaces, their curvatures satisfy the Yang-Mills equations [16, 17]. They stand nearer to the "skeleton" structure than any other spacetime. Other spacetimes — the Friedmann model, for example – will differ in two aspects: they will not be conformally flat, and their Lorentz bundles will have less symmetry.

Let us finish, however, by recalling a remarkable fact. In the favored Friedmann model with a cosmological constant, the Universe spends most of its lifetime in a radiation-dominated stage. It so happens that radiation does not contribute to the scalar curvature. The relation $\Lambda = -R/4$ and the above "algebraic" result consequently keep holding.

VI. SUMMING UP

Recent observation data point to a significant present-day value for the cosmological constant Λ. Added to the necessity of Λ-dominance at the earliest stages to provide for inflation, this leads to a picture of a Universe which starts as a de Sitter spacetime and is subsequently deformed by the effect of radiation and matter. Any spacetime is a differentiable manifold every point of which has for tangent space a Minkowski space, on which the Lorentz group acts. The whole structure is summed up in the bundle of Lorentzian frames on spacetime. On that bundle, the Lorentz generators are represented by vertical vector fields and the Minkowski vectors by horizontal fields. An analysis of the complete algebra of vector fields on the bundle shows that Λ is the eigenvalue of a Casimir operator of the Lorentz group.

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